[**Closed book, calculator, and notes**] Show all of your work clearly in the space provided or on the additional page at the end of the exam. If the additional page is used, clearly identify to which exam question it is related. Be sure to **read each problem carefully**. Note that the exam is double sided.

$$f(n) = \Theta(g(n)) \text{ and } g(n) = \Theta(h(n)) \quad \Rightarrow \quad f(n) = \Theta(h(n)) \tag{1}$$
$$f(n) = O(g(n)) \text{ and } g(n) = O(h(n)) \quad \Rightarrow \quad f(n) = O(h(n)) \tag{2}$$
$$f(n) = \Omega(g(n)) \text{ and } g(n) = \Omega(h(n)) \quad \Rightarrow \quad f(n) = \Omega(h(n)) \tag{3}$$

$$f(n) = \Theta(g(n)) \iff g(n) = \Theta(f(n)) \tag{4}$$

$$\lg n = \log_2 n \tag{5}$$
$$\ln n = \log_e n \tag{6}$$
$$a = b^{\log_b a} \tag{7}$$
$$\log_c(ab) = \log_c a + \log_c b \tag{8}$$
$$\log_b a^n = n \log_b a \tag{9}$$
$$\log_b a = \frac{\log_c a}{\log_c b} \tag{10}$$

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} = \Theta(n^2) \tag{11}$$

$$\sum_{k=0}^{n} x^k = 1 + x + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1} = \Theta(x^n), \qquad x \neq 1 \tag{12}$$

$$\sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \approx \ln n + .577 = \Theta(\log n) \tag{13}$$

Given positive functions $f(n)$ and $g(n)$ such that

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c$$

for some constant c.

1. If $0 < c < \infty$, then $f(n) = \Theta(g(n))$
2. If $0 \leq c < \infty$, then $f(n) = O(g(n))$
3. If $0 < c \leq \infty$, then $f(n) = \Omega(g(n))$

If $f(n)$ and $g(n)$ both approach zero or both approach ∞ in the limit, then

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

where $f'(n)$ and $g'(n)$ denote derivatives of f and g with respect to n.

1. (10 points) Is the following array a max-heap? Justify your answer.
A = ⟨23 17 14 6 13 10 1 5 7 12⟩

2. (10 points) Suppose we use a randomized version (the pivot is picked at random) of the Select algorithm discussed in class to select the minimum element of the array A = ⟨23 17 14 6 13 10⟩. Describe a sequence of partitions that result in a worst-case performance for our algorithm and identify the time complexity (using Θ notation) for the worst-case senario for an array of length n.

3. (10 points) Consider the following algorithm which accepts an array, A, of length n:

```
SelectionSort(A) {
    for i = 1 to A.length-1
        min = i
        for j = i+1 to A.length
            if A[j] < A[min]
                min = j
        swap A[i] with A[min]
```

Give an example input array, A, for which the SelectionSort algorithm is unstable. (Full credit will be given for the smallest possible array size.) Be sure to explain your example.

4. (10 points) Consider an array, A, of n elements already arranged in descending order. What is tight asymptotic time complexity for running insertion sort on A to produce an array in ascending order? Justify your answer.

5. Consider the following recurrence:

$$T(n) = \begin{cases} 2 & \text{if } n = 1, \\ 1 & \text{if } n = 2, \\ T(n-2) + T(2) + n & \text{if } n > 2. \end{cases}$$

(a) (15 points) Solve the recurrence, and be sure to state any assumptions you make.

(b) (5 points) Is this result an upper-bound, lower-bound, or tight bound? State the time complexity using the appropriate notation.

6. For each summation, show that it is either bounded or not bounded above by a constant.

(a) (10 points)

$$8 + \sum_{k=1}^{n} \frac{1}{k^2}$$

(b) (5 points)

$$8 + \sum_{k=-1}^{n+5} \frac{1}{k^2}$$

(c) (5 points)

$$8 + \sum_{k=-4}^{n-4} \frac{5}{(5+k)^2}$$

7. Given a set of n numbers, we wish to find the i largest in sorted order using a comparison-based algorithm. Analyze the running times of the algorithms described below in terms of n and i. Justify your answers.
(a) (10 points) Sort the numbers using Mergesort and list the i largest.

(b) (10 points) Find the largest of the first n elements, swap it with the last element, decrement n by one and repeat until $n = n - i$.

Additional work area for any problem. Clearly identify to which problem the work on this page is related.

Additional work area for any problem. Clearly identify to which problem the work on this page is related.